Segregation of granular media by diffusion and convection

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A diffusion-convection equation is used to model granular segregation within a mixture of particles of different size, shape, or surface structure in a vertical vessel. Convection describes competition between species in vertical direction whereas random noise (shaking) allows particles to exchange positions. For two species it is shown that the moving grains converge to a unique distribution along the vertical scale. For more than two species it is shown that at least one equilibrium distribution exists (there are examples with multiple equilibria). For a class of models with simple competition laws, uniqueness of the equilibrium in all dimensions is shown.

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I. INTRODUCTION

The Brazil nut effect is a well-known phenomenon in granular media: A mixture of two kinds of grains that differ in size is filled into a vertical (glass) cylinder and then stirred or shaken. The bigger grains tend to move up. In some sense the smaller particles fall between the gaps of the bigger ones. If the grains differ also in specific weight, shape, coarseness of the surface, etc., or if more than two kinds of grains are used, then the behavior may become more complex. It seems that there are not too many experimental results on segregation with more than two kinds or with a continuous distribution of grains.

However, the case of two types of particles has been studied by several authors. In [1–4] the particles are modeled as hard spheres. References [5,6] treat the kinetic theory of binary mixtures of spherical particles. In [3,4] a cooling process is used to describe segregation, whereas in [2] pattern formation caused by vibration is investigated. The behavior of rolling matter on the surface of a heap has been extensively studied (see, for example, [7,8]). In [9] an adaptation of the Monte Carlo method is used to analyze size segregation that occurs by shaking mixtures of two types of grains while the system is cooling, i.e., constantly losing energy. Cellular automaton models are used in [10–13] to study stratification and pattern formation in poured mixtures. In [14–18] continuum models are used to explain segregation in surface flows and flowing avalanches, i.e., so-called *kinetic sieving*. In [19] experiments and theory are compared with respect to surface flows in two-dimensional silos, and in [20] stratification is studied experimentally. References [21,22] deal with segregation and pattern formation in rotating drums.

Since the details of the interactions between several kinds of particles will differ widely and are also not generally known, we propose a heuristic model based on *diffusion* and *convection*, i.e., the model has the form of a diffusion-convection system. The idea is that the different species *compete* with each other for an appropriate position on a vertical scale via different convection rates and that random noise gives sufficient freedom for grains to pass between other grains.

The present model has some relation to the Kynch model for sedimentation (see [23]). However, the Kynch model does not contain a diffusion term. In particular, the simple competition dynamics studied here corresponds to the Masliyah dynamics in the Kynch model (see [24]).

The model studied in this paper is unrealistic insofar as we assume that the proportion of empty space is constant throughout the vessel independently of the composition of the material. Besides, the particles cannot be deformed. Hence we do not allow for compaction or compressibility. Thus, the system should be seen as an attempt towards a general model for segregation.

In Sec. II we describe the model system and we introduce the necessary invariance and conservation properties. In Sec. III we show that in the case of two species, there is a unique stationary distribution of grains that depends only on the initial total masses of the two species. This stationary solution is globally stable, i.e., for any initial distribution within the vessel (given the total masses) the solution of the diffusion-convection system will approach this equilibrium. In Sec. IV the case of more than two species is studied. We show that for any set of total masses there is at least one equilibrium solution. Simple examples with three species show that the equilibrium may not be unique. In Sec. V, we study a special model for n species with a simple competition law. Here we can prove uniqueness. Finally, Sec. VI gives a brief discussion of the results.

All proofs are deferred to the Appendix. In Sec. 1 of the Appendix we indicate why a pure convection model would not do.

II. THE MODEL

We consider a vessel in the form of a vertical cylinder of height l. We assume that the distribution of the material is homogeneous in the horizontal direction, i.e., we assume a one-dimensional model. We represent the vessel by the interval $[0,l]$ where $x=0$ corresponds to the bottom and $x=l$ to the top. We assume there are n types or species of particles numbered $i=1,\ldots,n$. Let $u_i(t,x)$ be the density of the ith species at level x and time t. We collect these into a vector $\mathbf{u}=(u_1,\ldots,u_n)^T$. Here $\mathbf{u}$ is a column vector, the symbol T means transpose. We assume that the motion of particles in the vertical direction, caused by shaking and the influence of gravity, can be described by diffusion and convection. Diffusion is a random (*Brownian*) undirected motion, convection is directed and depends strongly on the types of interacting particles and their relative densities. We understand the model in such a way that the diffusion term accounts for the random effects of shaking that provides space for convective motion, whereas the convection term incorporates effects of friction of grains and of small grains falling into the gaps between large grains ([25], [15]).

Let $J_i = J_i(\mathbf{u},x)$ denote the flux of species i at $x \in [0,l]$, i.e., the relative amount of particles of type i passing through an infinitesimal volume element per time. The equation governing the change in concentration of species i is then

$$\frac{\partial u_i}{\partial t} = -\frac{\partial J_i}{\partial x}. \tag{1}$$

For our purposes we split the flux $\mathbf{J}$ (with components J_i) into a diffusional part $\mathbf{J}^d$ and a convectional part $\mathbf{J}^c$. Thus Eq. (1) becomes $u_{it} = -(J^d_{ix} + J^c_{ix})$, where subscripts t and x denote partial derivatives with respect to space and time coordinates, respectively. Hence the model assumes the general form of a system of diffusion convection equations. We underline that this is a standard form of model that should be applied when a more detailed description (transport equation, Boltzmann equation, particle model) is not available or not applicable for lack of estimates for experimental parameters. In fact, for most detailed models diffusion-convection equations occur as limiting cases for rapid motion and frequent changes of direction. The model must be written in divergence form because of conservation of mass.

If we choose $J_i^d = -D_i(\mathbf{u}) u_{ix}$ and $J_i^c = f_i(\mathbf{u})$ for some vector-valued function $\mathbf{f}$ with components f_i, the model assumes the general form

$$u_{it} = [D_i(\mathbf{u}) u_{ix} - f_i(\mathbf{u})]_x, \quad i = 1, \ldots, n.$$

We assume that the functions D_i and f_i are twice continuously differentiable. We collect the diffusion coefficients into a diagonal matrix $D(\mathbf{u}) = [D_i(\mathbf{u})\delta_{ij}]$ and we interpret the functions $f_i(\mathbf{u})$ as components of a vector field $\mathbf{f}(\mathbf{u})$. In vector notation the system assumes the form

$$\mathbf{u}_t = [D(\mathbf{u})\mathbf{u}_x - \mathbf{f}(\mathbf{u})]_x. \tag{2}$$

The model does not account for empty space between grains. It is tacitly assumed that the grains fill the volume completely or, equivalently, that empty space is evenly distributed throughout the vessel whatever the distribution of species is. Although this assumption is not realistic, the model is a step toward the study of mixtures. More complex models would allow for variable distributions of empty space and hence also for compaction effects.

System (2) must be supplied with boundary conditions. As we shall see in a moment, the requirement of conservation of total mass for each species determines the boundary condition uniquely. We require

$$D(\mathbf{u})\mathbf{u}_x - \mathbf{f}(\mathbf{u}) = \mathbf{0} \quad \text{at} \quad x=0 \quad \text{and} \quad x=l. \tag{3}$$

In order that the model be realistic, we have to impose two further requirements: (i) the particle densities u_i are nonnegative, (ii) at each space point x and time t the particle densities add up to 1.

For these requirements to be fulfilled the following assumptions on the diffusion rates and the vector field are sufficient and also necessary.

(a) The diffusion rates (which may depend on the vector $\mathbf{u}$ of densities) are the same for all species, i.e., the diffusion matrix $D(\mathbf{u})$ is a multiple $d(\mathbf{u})I$ of the identity matrix.

(b) The diffusion rate is positive, $d(\mathbf{u}) > 0$.

(c) $\Sigma_{i=1}^n f_i(\mathbf{u}) \equiv 0$. This ensures that particle densities always add up to one.

(d) $u_i = 0$ implies $f_i(\mathbf{u}) = 0$. This guarantees that concentrations cannot become negative.

(e) $\partial f_i / \partial u_j |_{u_i=0} = 0$ for $i \neq j$.

With these hypotheses, the system and the boundary condition read

$$\mathbf{u}_t = [d(\mathbf{u})\mathbf{u}_x - \mathbf{f}(\mathbf{u})]_x, \tag{4}$$

$$d(\mathbf{u})\mathbf{u}_x - \mathbf{f}(\mathbf{u}) = \mathbf{0} \quad \text{at} \quad x=0 \ \text{and} \ x=l. \tag{5}$$

The positivity of the diffusion coefficient [condition (b)] is just the standard condition that ensures that diffusion is not degenerate. In the Appendix we show that the conditions (a)–(e) indeed yield conservation of positivity and mass, as requested.

The general case of n species leads deeply into the qualitative analysis of diffusion convection equations, and perhaps we need further insights into the mechanics of segregation to choose the right functions $\mathbf{f}$. Therefore, we proceed to the most important special case of two species and later we return to a special equation for $n \geqslant 2$ species of grains.

III. THE CASE OF TWO SPECIES

In the case of two species we can, in view of $\Sigma u_i = 1$, put $u_1 = u$, $u_2 = 1-u$ and $f_1(u_1,u_2) = f_1(u,1-u) = f(u)$. Then we get the scalar diffusion-convection equation

$$u_t = [d(u)u_x - f(u)]_x \tag{6}$$

with the boundary condition

$$d(u)u_x - f(u) = 0 \ \text{at} \ x=0 \ \text{and} \ x=l. \tag{7}$$

As in the general case, we require conditions (a)–(e) above. In the case $n=2$ these assume the following simpler form: (a),(b) $d(u) > 0$; (c) says that $f_2 = -f_1$, hence becomes redundant; (d) $f(0) = f(1) = 0$. (e) is a consequence of (d) for $n=2$.

First we look for stationary solutions. From Eq. (6) we get $[d(u)u_x - f(u)]_x = 0$, hence $d(u)u_x - f(u)$ is a constant that vanishes in view of the boundary condition (7). Hence stationary solutions correspond to solutions of the ordinary differential equation

$$\frac{d}{dx} u = g(u) \tag{8}$$

where

$$g(u)=f(u)/d(u). \tag{9}$$

Of course we have

$$g(0)=g(1)=0 \tag{10}$$

by condition (d). Before investigating the qualitative behavior of the model we discuss what we can expect. In the case of only two species, i.e., in the case of one scalar differential equation, the process of reshuffling by convection and diffusion should lead to a stationary distribution. However, since we can think of many different initial distributions with different proportions of species, there must be many stationary solutions. But we can show that for any given proportion of the total masses of the two species there is exactly one equilibrium. This fact is expressed in the first proposition.

Proposition 1. Let $\overline{u}l$ be the total amount of the first species in the vessel,

$$\int_0^l u(x)dx=\overline{u}l. \tag{11}$$

For any number $\overline{u}\in[0,1]$ the differential equation (8) has exactly one solution satisfying condition (11).

The proof is given in the Appendix. Next we discuss the behavior of the time-dependent problem, Eqs. (6) and (7). In the Appendix we show the following result.

Proposition 2. For any initial distribution of species 1 the time-dependent solution of Eqs. (6) and (7) converges to the unique equilibrium distribution characterized by the total proportion $\overline{u}$ of species 1.

As an example we choose $d(u)\equiv D=\text{const}$ and

$$f(u)=u(1-u), \tag{12}$$

which corresponds to $f_1(u_1,u_2)=-f_2(u_1,u_2)=u_1u_2$, thus saying that whenever two particles of type 1 and 2 interact, the type 1 particle is pushed up and the type 2 particle is pushed down. In Sec. V it is shown that this dynamics is equivalent to the competitive dynamics for $n=2$ and a special choice of the parameters. Hence we can think of the type 2 particles as having Stokes's settling velocity with respect to the type 1 particles. Figure 1 shows the typical behavior of the species distribution for a function f of type (12). Solutions for very small and very large proportion of u_1 are convex or concave, respectively, intermediate solutions show *arctan* shapes. The figure shows clearly that the solutions of the ordinary differential equation can be parametrized either by their initial data, i.e., their values at $x=0$, or by the total mass $\overline{u}$. If we use Eq. (12) in our model, the ordinary differential equation (8),

$$u'=\frac{1}{D}u(1-u), \tag{13}$$

is a Riccati equation (for details see [26]). The solution for the initial datum $u^0\in[0,1]$ is

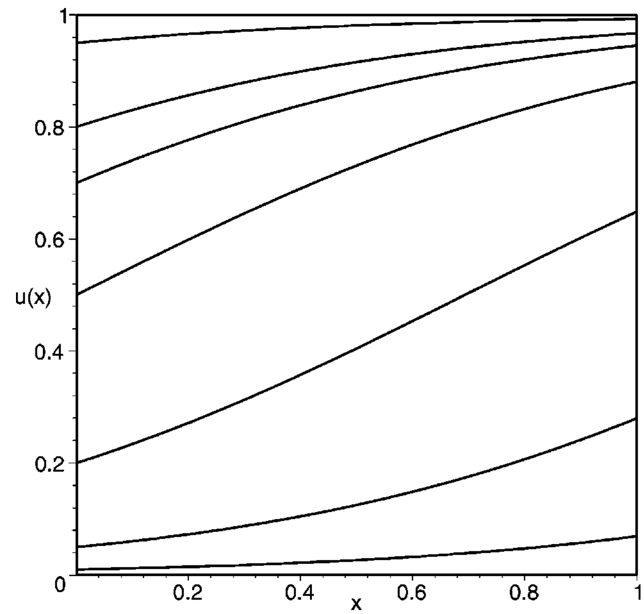

FIG. 1. The layers of stationary solutions for $f(u)=u(1-u)$, $d(u)\equiv 1$, and $l=1$. Each curve represents the local proportion of the first one of two species as a function of the height $x\in[0,1]$ in the vessel. x has units of length, whereas $u(x)$ represents a relative concentration and is thus dimensionless. The family of curves is parametrized either by the initial value or by the total mass.

$$u(x)=\frac{u^0}{(1-u^0)e^{-x/D}+u^0},$$

which is represented in Fig. 2. We observe that in the stationary solution the first substance u_1 is concentrated at the bottom of the vessel whereas u_2 has its maximum concen-

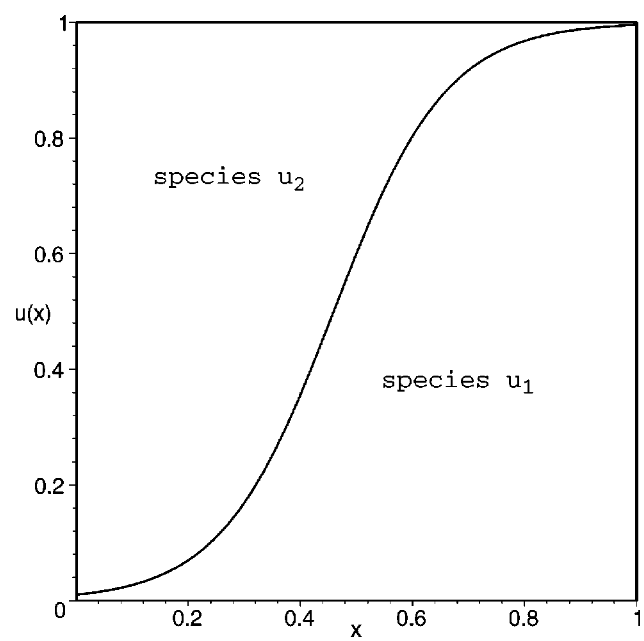

FIG. 2. Segregation of two types of particles due to diffusion and convection. System (6) with $d(u)\equiv D=0.1$ and $u(0)=0.01$. x has dimensions of length, $u(x)$ is dimensionless.

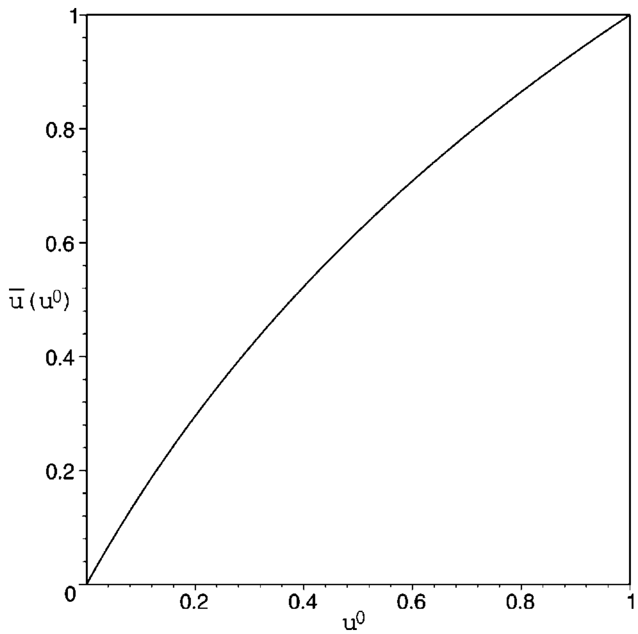

FIG. 3. The total mass of species u as a function of the initial value u^0. Parameter values $D=1$ and $l=1$. The resulting map is a bijection of the interval $[0,1]$. Both coordinates represent relative concentrations without units.

tration at the top. We give the exact expression for the total mass of species $u=u_1$ as a function of the initial value u^0 as

$$\int_0^l u(x)dx=\int_0^l \frac{u^0}{(1-u^0)e^{-x/D}+u^0}dx = D\cdot \ln[(1-u^0)+u^0 e^{l/D}]. \tag{14}$$

The dependence of $\overline{u}$ on u^0 is shown in Fig. 3. We emphasize at this point that the example discussed in this section in a certain sense characterizes the qualitative behavior of *any* system of the form (6). In a general situation some profiles may be increasing, some decreasing, but they always form a set of nonintersecting layers.

At the end of the section we briefly discuss the general case of more than two types of grains, thus underlining the particular nature of the case $n=2$. For $n\geqslant 3$ we cannot rule out that the process of reshuffling by convection and diffusion leads to rather complicated dynamics as, for example, oscillatory behavior. One could imagine as well that the system approaches different stationary solutions with the same proportion of total masses depending on the initial distribution. Such phenomena do not occur for $n=2$. Stationary solutions can still be parametrized by their initial data (as ensured by the existence and uniqueness theorem for ordinary differential equations), but there may be several solutions for a given vector of proportions of total masses. A more detailed analysis of stationary solutions in the general case is reported in the following section.

IV. STATIONARY SOLUTIONS IN THE CASE OF n SPECIES

The stationary solutions of system (2) with boundary conditions (3) are solutions of the system of ordinary differential equations

$$\dot{\mathbf{u}}=\mathbf{g}(\mathbf{u}), \tag{15}$$

where $\mathbf{g}(\mathbf{u})=\mathbf{f}(\mathbf{u})/d(\mathbf{u})$. Let $\mathbf{G}$ be the solution operator of (15), i.e.,

$$\frac{d}{dt}\mathbf{G}(t,\mathbf{u})=\mathbf{g}(\mathbf{G}(t,\mathbf{u})),$$

$$\mathbf{G}(0,\mathbf{u})=\mathbf{u}.$$

From hypotheses (c) and (d) it follows that the function $\mathbf{g}$ satisfies

$$u_i=0\Rightarrow g_i(\mathbf{u})=0, \tag{16}$$

$$\sum g_i(u)\equiv 0. \tag{17}$$

Define the simplex (generalized triangle or tetrahedron) of probability vectors

$$S=\left\{\mathbf{u}\in\mathbb{R}^n | u_i\geqslant 0, \quad i=1,\ldots,n, \quad \sum_{i=1}^n u_i=1\right\}.$$

The set S is the set of local distributions of the n species. The conditions (16) and (17) ensure that solutions of Eqs. (15) starting in S remain in S for all times t (positive or negative). Hence S is invariant with respect to the flow of Eq. (15). If $\mathbf{G}(t,\mathbf{u})$, $0\leqslant t\leqslant l$ is the solution of Eq. (15) starting from $\mathbf{u}(0)=\mathbf{u}$, then

$$\int_0^l \mathbf{G}(t,\mathbf{u})dt=\overline{\mathbf{u}}l \tag{18}$$

defines the vector $\overline{\mathbf{u}}$ of total masses of the n species. The next theorem shows that for any distribution $\overline{\mathbf{u}}\in S$ of total masses there is *at least one* stationary solution (exactly one in the case $n=2$ by Proposition 1).

Theorem 1 Assume Eqs. (16) and (17). For every $\overline{\mathbf{u}}\in S$ there is at least one $\mathbf{u}\in S$ such that Eq. (18) holds.

The proof is given in the Appendix. In general there will be more than one stationary solution for a given mass distribution. This can be seen from the following argument. Assume the system (15) has a nonconstant periodic solution with minimal period $\omega>0$. Choose $l=\omega$. Then for all points $\mathbf{u}$ on the periodic orbit we get the same value $\overline{\mathbf{u}}$. In this example we can extend the function from $[0,l]$ periodically to all of the real axis to get an infinitely high vessel with a periodic distribution of species. Then the phase of this distribution can be chosen arbitrarily. We do not claim that such a choice of the function $\mathbf{f}$ or $\mathbf{g}$ is realistic.

We reformulate the result in terms of boundary value problems of second order systems of ordinary differential equations. Theorem 1 says that the boundary value problem

$$\ddot{\mathbf{v}}=\mathbf{g}(\dot{\mathbf{v}}),\quad \mathbf{v}(0)=\mathbf{0},\quad \mathbf{v}(l)=\overline{\mathbf{u}}l$$

has at least one solution in S for every $\overline{\mathbf{u}}\in S$.

V. A SIMPLE COMPETITION LAW

There is one choice of the vector field $\mathbf{f}$ for which we can show uniqueness of the stationary state in any dimension n. Here we assume that particles of the ith species have a *preferred* velocity m_i and that their actual velocity is defined by the balance of mass. In other words, the actual velocity at each point is the difference of its preferred velocity and the mean velocity of all particles at that space point. We choose $D=\text{const}$ and

$$\mathbf{f}(\mathbf{u})=M\mathbf{u}-\left[\sum_{i=1}^{n}(M\mathbf{u})_i\right]\mathbf{u}, \tag{19}$$

where $M=(m_i\delta_{ij})$ is a diagonal matrix of order n and the m_i are strictly positive constants. Without loss of generality we assume $m_1>m_2>\cdots>m_n$. In coordinate notation the ordinary differential equation (15) reads

$$\frac{d}{dx}u_i=\frac{1}{D}\left[m_iu_i-\left(\sum_{k=1}^{n}m_ku_k\right)u_i\right]. \tag{20}$$

This equation is just a renormalization of the linear system $\dot{\mathbf{u}}=M\mathbf{u}$. In theoretical ecology this type of system has been called the *replicator equation* (for an overview on replicator equations see for example, [27]). The solution of Eq. (20) can be explicitly given as

$$u_i(x)=\frac{e^{m_ix/D}u_i(0)}{\sum_{j=1}^{n}e^{m_jx/D}u_j(0)}. \tag{21}$$

Then the equation

$$\int_0^l\frac{e^{m_ix/D}u_i}{\sum_{j=1}^{n}e^{m_jx/D}u_j(0)}dx=l\overline{u_i} \tag{22}$$

establishes the connection between stationary solutions and total masses.

Theorem 2. Consider the system (2) with boundary condition (3) where the nonlinearity is given by Eq. (19). For any choice $\overline{\mathbf{u}}\in S$ of the total masses there is exactly one stationary solution.

The proof is given in the Appendix.

If we choose $n=2$ then $f_1(u_1,u_2)=m_1u_1-(m_1u_1+m_2u_2)u_1=(m_1-m_2)u_1(1-u_1)$. Hence we get essentially Eq. (12), up to the constant factor m_1-m_2.

VI. DISCUSSION

The diffusion-convection model discussed in this paper describes segregation of particles of varying size and physical qualities such as specific density of the material or surface structure in a vertical vessel. In the case of only two *competing* species the dynamics are predicted to be relatively simple. Stationary solutions are monotone functions, and every time-dependent solution eventually reaches a stable state. In the general case of three or more species the dynamics can exhibit far more complicated behavior. Stationary states are in general not monotone any more and there can be different stationary solutions for a given proportion of total masses in the vessel. However, for every given proportion of total masses there is at least one stationary distribution. Uniqueness can be shown under relatively strong conditions on the convection function $\mathbf{f}$. An important example is the competitive kinetics corresponding to the Masliyah kinetics in sedimentation models or the so-called *replicator kinetics* in biological models. Our model is unrealistic insofar as it does not account for compaction or compressibility effects.

ACKNOWLEDGEMENT

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APPENDIX

1. The hyperbolic problem

If we consider a system of the form of Eq. (2) without diffusion, i.e., a model with pure convection, then we get a totally different problem. For example, consider the case $n=2$. Equation (6) becomes the conservation law

$$u_t+f(u)_x=0. \tag{A1}$$

The characteristic differential equations (with parameter s) are $\dot{t}=1$, $\dot{x}=f'(u)$, $\dot{u}=0$, hence the characteristic curves are straight lines $x=x_0+f'(u_0)(t-t_0)$. If the function $u_0(x)$ is not monotone, then the behavior can be very complicated. If, however, we let $D(u)$ go to zero, then we get a *viscosity solution*. In particular, the stationary solution becomes a step function.

2. Conservation of mass and positivity

From the differential equation (2) and the boundary condition (3) we get

$$\frac{d}{dt}\int_0^l\mathbf{u}dx=\int_0^l[D(\mathbf{u})\mathbf{u}_x-\mathbf{f}(\mathbf{u})]_xdx=[D(\mathbf{u})\mathbf{u}_x-\mathbf{f}(\mathbf{u})]\big|_0^l=\mathbf{0}. \tag{A2}$$

Hence total mass is preserved. Next, in view of Eqs. (4) and (5) and hypothesis (c), the scalar function Σu_i satisfies the diffusion equation

$$\left(\sum u_i\right)_t = \sum u_{it} = \left[d(\mathbf{u})\left(\sum u_i\right)_x\right]_x \tag{A3}$$

and the boundary condition

$$\left(\sum u_i\right)_x(0) = \left(\sum u_i\right)_x(l) = 0. \tag{A4}$$

We have $\Sigma u_i^0(x) \equiv 1$. Hence both the given function u and the function $\equiv 1$ satisfy the differential equation (A3) and the boundary condition (A4), and therefore they are identical, $\Sigma u_i(t,x) \equiv 1$. This shows that everywhere masses add up to 1.

Now we sketch the proof that positivity is preserved. The differential equation (2) reads in component notation

$$u_{it} = [d'(\mathbf{u}) \cdot \mathbf{u}_x] u_{ix} + d(\mathbf{u}) u_{ixx} - [f_i'(\mathbf{u}) \cdot \mathbf{u}_x].$$

Notice that the brackets are inner products. Suppose that at time $t = \bar{t}$ the ith component u_i vanishes for the first time at $x = \bar{x} \in [0,l]$. First assume $0 < \bar{x} < l$. Then $u_{ix}(\bar{t},\bar{x}) = 0$, $u_{it}(\bar{t},\bar{x}) \leqslant 0$, $u_{ixx}(\bar{t},\bar{x}) \geqslant 0$. Hence at this point

$$u_{it} = d(\mathbf{u}) u_{ixx} - \sum_{j \neq i} \frac{\partial f_i}{\partial u_j} \frac{\partial u_j}{\partial x}.$$

The left-hand side is nonpositive, the first term on the right-hand side is non-negative, and the second term on the right-hand side vanishes in view of property (e). Hence we would have a contradiction if one of the inequalities were strict. This argument shows how property (e) enters. Next assume $\bar{x} = 0$ (or $\bar{x} = l$). Then $\mathbf{u}(\bar{t},x)$ goes to zero from positive values, and so does $\mathbf{u}_x(t,x)$ in view of the boundary condition. It seems that it is *unlikely* that $\mathbf{u}$ and $\mathbf{u}_x$ pass through zero at the same time.

There is a well-developed machinery for maximum and comparison theorems for parabolic equations and for getting strict inequalities from weak inequalities by using comparison functions (see [28–30]). It appears that the results in these monographs apply only to standard Dirichlet or Neumann conditions or to boundary conditions of the third kind with strong monotonicity properties. For our type of boundary condition we had to use a specially designed comparison function and some local estimates.

3. Proof of Proposition 1

We note that condition (10) ensures that every solution $u(x)$ of Eq. (8) with initial condition $u^0 = u(0) \in M := [0,1]$ never leaves M. By the existence and uniqueness theorem for ordinary differential equations, Eq. (8) has exactly one solution for each $u^0 \in M$. By the same argument solution curves cannot intersect and thus lie one above each other in the way indicated in Fig. 1. Each solution is either increasing or decreasing or constant. In other words, the mapping $F: u^0 \mapsto \bar{u}$ is strictly monotone and therefore one to one. Furthermore, the solutions of Eq. (8) depend continuously on the initial data u^0 and we have $F(0) = 0$, $F(1) = l$ in view of Eq. (10). As a consequence, the mapping F is onto.

4. Proof of Proposition 2

Define the function of two variables (x,t),

$$v := d(u) u_x - f(u).$$

Then

$$v_t = d'(u) u_t u_x + d(u) u_{xt} - f'(u) u_t,$$

$$v_x = u_t,$$

and thus

$$v_t = a(t,x) v_{xx} + b(t,x) v_x \tag{A5}$$

with $a(t,x) = d(u(t,x))$, $b(t,x) = d'(u) u_x - f'(u)$. The coefficients a and b are bounded. Clearly, v vanishes on the boundary $\{0,l\}$ in view of Eq. (7). With the smoothness conditions requested, the solution v goes to zero together with its space derivative ([28], p. 158, Theorem 1).

We point out the following fact: The function $v = d(u) u_x - f(u)$ satisfies Eq. (A5). Suppose at time $t = 0$

$$d(u) u_x \geqslant f(u) \tag{A6}$$

[or $d(u) u_x \leqslant f(u)$]. Then by the comparison principle, the same inequality holds for all positive t. If, for example, f is non-negative (see, for example, Sec. III) and Eq. (A6) holds initially, then u will be monotone in x for all time.

5. Proof of Theorem 1

Under the hypotheses of Sec. II the system lives on the simplex S. In the following argument we use the topological degree (see [31]). Let L be some positive number (the largest height of the vessel we are interested in). Define the function $\mathbf{H}: [0,L] \times \mathbb{R}^n \to \mathbb{R}^n$ by

$$\mathbf{H}(l,\mathbf{u}) = \frac{1}{l} \int_0^l \mathbf{G}(s,\mathbf{u}) ds \quad \text{for} \quad l > 0, \tag{A7}$$

$$\mathbf{H}(0,\mathbf{u}) = \mathbf{u}. \tag{A8}$$

The function $\mathbf{H}$ is continuous and it maps $[0,L] \times S$ into S, $H(0,\cdot)$ is the identity.

Let $\mathbf{u} \in \partial S$, i.e., $u_i = 0$ for some i. Then $G_i(l,\mathbf{u}) = 0$ for the same i and hence also $H_i(l,\mathbf{u}) = 0$. Conversely, let $\mathbf{H}(l,\mathbf{u}) \in \partial S$ for some l and $\mathbf{u}$. Then $H_i(l,\mathbf{u}) = 0$ for some i and hence $\int_0^l G_i(t,\mathbf{u}) dt = 0$ or, equivalently, $G_i(t,\mathbf{u}) = 0$ for all $t \in [0,l]$. Then $u_i = 0$, which means $\mathbf{u} \in \partial S$.

In short, $H(l,\cdot)$ maps the interior $S°$ of S into the interior and the boundary ∂S into itself. Even each lower-dimensional face of the boundary is mapped into itself.

Now let $\bar{\mathbf{u}} \in S°$. Then $\mathbf{H}(l,\mathbf{u}) \neq \bar{\mathbf{u}}$ for all $\mathbf{u} \in \partial S$. Thus, $\mathbf{H}$ defines a homotopy of S and the degree deg$(\mathbf{H}(l,\cdot),S,\bar{\mathbf{u}})$ is well defined for all $l \in [0,L]$. Since deg$[\mathbf{H}(0,\cdot),S,\bar{\mathbf{u}}] = 1$, we have deg$[\mathbf{H}(l,\cdot),S,\bar{\mathbf{u}}] = 1$ for all $l \in [0,L]$ and hence the equation $\mathbf{H}(l,\mathbf{u})=\overline{\mathbf{u}}$ has at least one zero in $\mathbf{u}\in S$ for all $l \in[0,L]$ (and hence for all positive l).

Now assume $\overline{\mathbf{u}}\in \partial S$. Then $\overline{u_i}=0$ for some i. Then, in the above argument, replace S by the face of S corresponding to the positive components of $\overline{\mathbf{u}}$.

In the general case $n\geqslant 3$ uniqueness does not hold.

6. Proof of Theorem 2

We can assume $D=1$. In this case the function $\mathbf{H}(l,\cdot)$, for $l>0$, is given componentwise by

$$H_i(l,\mathbf{u})=\frac{1}{l}\int_0^l \frac{e^{m_i x}u_i}{\sum_{j=1}^{n} e^{m_j x}u_j}dx. \tag{A9}$$

$\mathbf{H}(l,\cdot)$ maps S onto S (as shown in the proof of Theorem 4.1). Of course $\mathbf{H}(0,\mathbf{u})=\mathbf{u}$ on S as before.

Now we want to compute the Jacobian of $\mathbf{H}(l,\cdot)$ at a point $\mathbf{u}\in S$. For that purpose we interpret Eq. (A9) as a mapping from $\mathbb{R}^n\rightarrow\mathbb{R}^n$ and later restrict to the set S. The Jacobian J has elements

$$\frac{\partial H_i}{\partial u_i}=\frac{1}{l}\int_0^l \frac{1}{\left(\sum_{k=1}^{n} e^{m_k x}u_k\right)^2} e^{m_i x}\left(\sum_{k\neq i} e^{m_k x}u_k\right)dx>0,$$

$$\frac{\partial H_i}{\partial u_j}=-\frac{1}{l}\int_0^l \frac{e^{(m_i+j_j)x}u_i}{\left(\sum_{k=1}^{n} e^{m_k x}u_k\right)^2}dx<0, \quad i\neq j.$$

Thus, J has positive diagonal elements and negative off-diagonal elements. We have $\Sigma H_i(l,\mathbf{u}) =(1/l)\int_0^l[\Sigma G_i(t,\mathbf{u})]dt=0$. Hence the row vector $\mathbf{e}:=(1, \ldots ,1)$ is a left eigenvector with eigenvalue 0 and this eigenvector is perpendicular to the simplex S. The weak Hadamard criterion (or the Perron-Frobenius theorem for positive matrices) ensures that $\lambda=0$ is a simple eigenvalue. Hence the restriction of $\mathbf{H}$ to the simplex S has a Jacobian J with full rank $n-1$ and $\mathbf{H}(l,\cdot)$ is a diffeomorphism of S.

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